Fourier-Walsh coefficients for a coalescing flow (discrete time)

Boris Tsirelson

Abstract

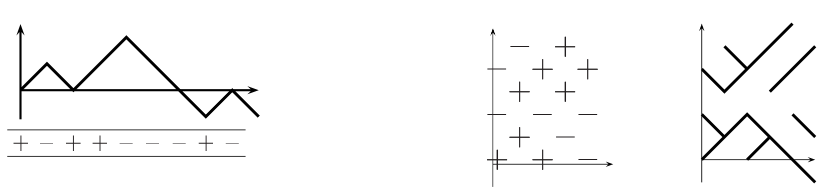
A two-dimensional array of independent random signs produces coalescing random walks. The position of the walk, starting at the origin, after n steps is a highly nonlinear, noise sensitive function of the signs. A typical term of its Fourier-Walsh expansion involves the product of $\sim \sqrt{n}$ signs.

Introduction

The simple random walk is driven by a one-dimensional array of independent random signs (Fig. 1a). The walk is a *stable* function of the signs, in the sense that changing at random a small fraction of the signs results in a small change of the walk.

A two-dimensional array of independent random signs may be used for producing the simple system of coalescing random walks (Fig. 1b). Changing at random a small fraction of the signs causes a dramatic change of the walks (Fig. 2). These are *sensitive* functions of the signs.

Stability and sensitivity, introduced by Benjamini, Kalai, Schramm [1], can be formulated also in terms of the Fourier-Walsh transform. Any function of random signs can be written as a polynomial, each term being a product



(a) (b)

Figure 1: (a) One-dimensional array of random signs produces a random walk. (b) Two-dimensional array of random signs produces coalescing walks.

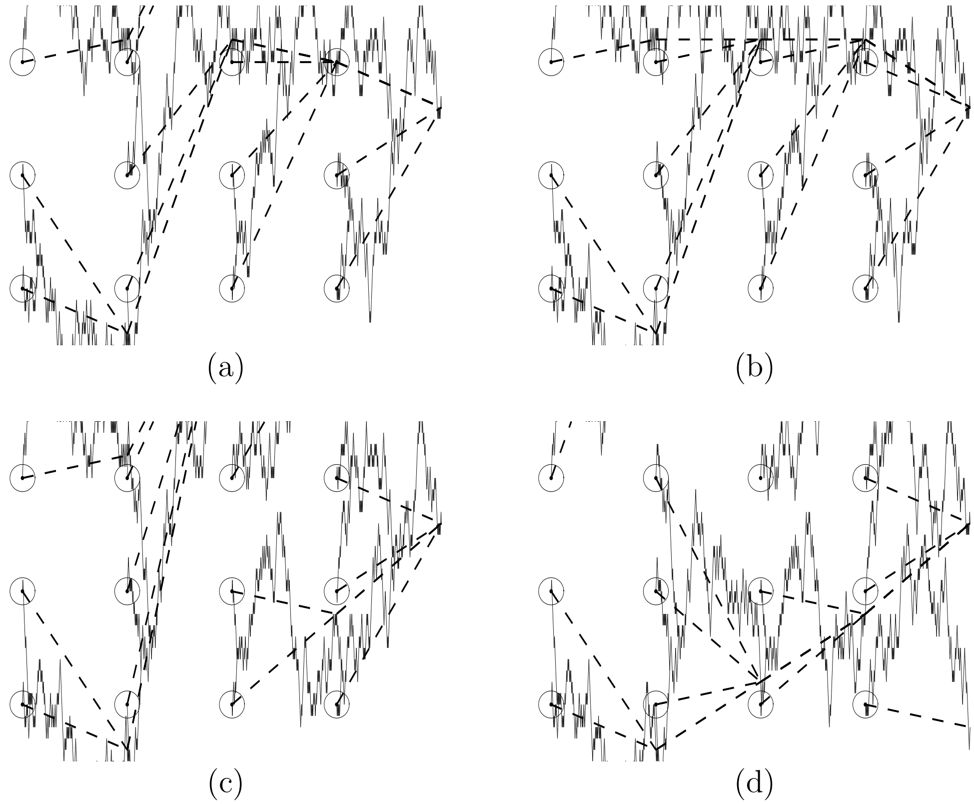


Figure 2: Noise sensitivity. Coalescing walks on the grid 1000×30, starting from $4 \times 3 = 12$ points (circled). Unperturbed array of random signs (a). Perturbed array: each random sign is flipped with probability 0.025 (b). Further perturbation of the same type (c). Still further (d).

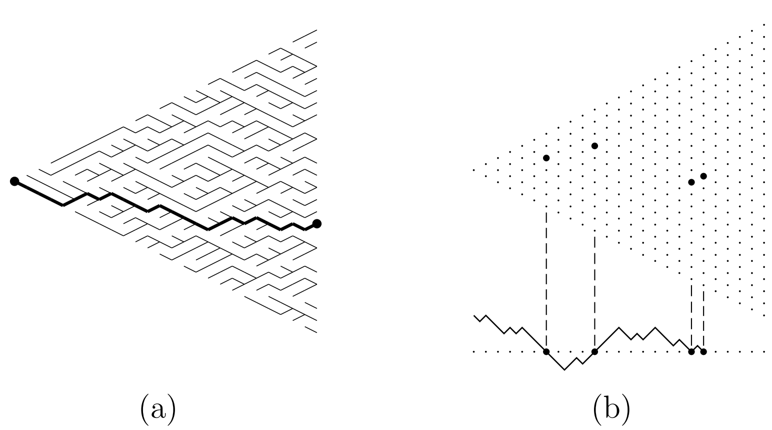


Figure 3: The function of a triangular array of random signs (a), and its typical spectral set (b).

of several signs, with a coefficient. These are Fourier-Walsh coefficients. For a stable function, they are concentrated on low degrees (frequencies), for a sensitive function — on high frequencies.

The simplest nontrivial function of coalescing walks is the final position of the walk starting at the origin (Fig. 3a). Its Fourier-Walsh transform is studied here. A typical term of the polynomial appears to involve about $\sqrt{n}$ signs (which is more than enough for sensitivity). The corresponding set on the time axis is distributed like the set of zeros of the simple random walk in twice faster time (Fig. 3b). Implications for continuous time will be published separately.

1 The model and results

Choose $n \in \{1, 2, \dots\}$ and introduce a triangular array of independent random signs

$$\tau(x, y), \quad x = 0, 1, \dots, n-1, \quad y = -x, -x+2, \dots, x-2, x,$$

each $\tau(x, y)$ being -1 or $+1$ with probabilities $1/2$. Define $W(x)$ for $x = 0, 1, \dots, n$ by

$$W(0) = 0; \quad W(x+1) = W(x) + \tau(x, W(x)) .$$

Clearly, $W(\cdot)$ is a simple random walk. In particular, $\mathbb{E}\, W(n) = 0$ and $\mathbb{E}\, W^2(n) = n$. We consider the Fourier-Walsh transform of the function $\xi = n^{-1/2} W(n)$ of our random signs $\tau(x, y)$:

$$\begin{gathered}
\frac{1}{\sqrt{n}} W(n) = \sum_{S \subset I} \hat{\xi}(S) \tau(S) , \\
I = \{ (x,y) : x = 0, 1, \dots, n-1, \; y = -x, -x+2, \dots, x-2, x \} , \\
\tau(S) = \prod_{(x,y) \in S} \tau(x, y) , \quad \hat{\xi}(S) = \mathbb{E} \big(\tau(S) \xi \big) = \frac{1}{\sqrt{n}} \mathbb{E} \big(\tau(S) W(n) \big) .
\end{gathered}$$

The equality $\mathbb{E}\, \xi = 0$ becomes $\hat{\xi}(\emptyset) = 0$, while $\mathbb{E}\, \xi^2 = 1$ becomes $\sum_{S \subset I} |\hat{\xi}(S)|^2 = 1$, since $\tau(S)$ are an orthonormal basis of $l_2\big(\{-1, +1\}^I\big)$.

....

1.1 Proposition $\hat{\xi}(S) = 0$ unless S is of the form

$$\begin{aligned}
& S = \{ (x_1, y_1), \dots, (x_m, y_m) \} , \\
& m \in \{1, \dots, n\} , \\
& 0 \le x_1 < \dots < x_m < n , \qquad (1.2) \\
& |y_1| \le x_1 , \\
& |y_{k+1} - y_k| \le x_{k+1} - x_k \quad \text{for } k = 1, \dots, m-1 ,
\end{aligned}$$

in which case

$$\hat{\xi}(S) = \frac{1}{\sqrt{n}} p(x_1, y_1) q(S)\,,$$

where

$$\begin{aligned} q(S) &= \\ &= \prod_{k=1}^{m-1} \frac{p(x_{k+1}-x_k-1, y_{k+1}-y_k-1) - p(x_{k+1}-x_k-1, y_{k+1}-y_k+1)}{2}\,, \\ p(x,y) &= 2^{-x} \frac{x!}{\left(\frac{x-y}{2}\right)!\,\left(\frac{x+y}{2}\right)!}\,. \end{aligned}$$

. . . .

Define a probability distribution μ_ξ (the spectral measure of ξ) on the set $2^I = \{S : S \subset I\}$ by

$$\mu_\xi(A) = \sum_{S \in A} |\hat{\xi}(S)|^2 \quad \text{for } A \subset 2^I\,.$$

Let S^{random} be a random subset of I, distributed according to μ_ξ. We know that S^{random} is of the form (1.2) with probability 1. The projection of S^{random} onto the first axis is another random set $R^{\text{random}} = \{x_1, \dots, x_m\}$. The following result describes the probability distribution of R^{random}.

. . . .

1.3 Proposition Whenever $m \in \{1, \dots, n\}$ and $0 \le x_1 < \dots < x_m < n$,

$$\begin{aligned} \mathbb{P}\big(R^{\text{random}} = \{x_1, \dots, x_m\} \big) &= \\ &= \frac{1}{n} p(2x_1, 0) \cdot \prod_{k=1}^{m-1} \big(p(2x_{k+1} - 2x_k - 2, 0) - p(2x_{k+1} - 2x_k, 0) \big)\,. \end{aligned}$$

. . . .

It means that R^{random} may be described as follows. First, we choose its maximal element x_m (without knowing m for now) uniformly on $\{0, \dots, n-1\}$. Then we introduce a simple random walk $V = \big(V(0), V(1), \dots\big)$ and consider $V(2(x - x_m))$; that may be thought of as a random walk starting at $(x_m, 0)$ and going backward in time, twice faster than usual, see Fig. 3b. Its zeros are just R^{random},

$$R^{\text{random}} = \{x \in \{0, \dots, x_m\} : V(2(x - x_m)) = 0\}\,. \tag{1.4}$$

It is strange! After proving Prop. 1.3 we may introduce V satisfying (1.4). It would be more natural to do other way round.

1.5 Problem Can Prop. 1.3 be deduced from (1.4)? That is, can V be introduced somehow before proving Prop. 1.3?

2 Proofs

The random path $W = \big(W(0), \dots, W(n)\big)$ determines a partition of our probability space $\Omega = \{-1, +1\}^I$ into subsets $\{W = w\}$ indexed by paths w, that is, by sequences $w = \big(w(0), \dots, w(n)\big)$ such that $w(0) = 0$ and $w(k+1) - w(k) = \pm 1$ for $k = 0, \dots, n-1$.

Proof of Prop. 1.1.
We have $\hat\xi(S) = \mathbb{E}\left(\tau(S)\xi\right) = \sum_w \mathbb{E}\left(\tau(S)\xi\mathbf{1}_{(W=w)}\right)$, where $\mathbf{1}_{(W=w)}$ is the indicator of the event $W = w$. However, $W(n)\mathbf{1}_{(W=w)} = w(n)\mathbf{1}_{(W=w)}$, thus $\hat\xi(S) = n^{-1/2}\sum_w w(n)\,\mathbb{E}\left(\tau(S)\mathbf{1}_{(W=w)}\right)$. Clearly, $\mathbf{1}_{(W=w)}$ is independent of all $\tau(x,y)$ with $w(x) \ne y$. Therefore $\mathbb{E}\left(\tau(S)\mathbf{1}_{(W=w)}\right) = 0$ unless w passes through all points of S, which proves (1.2).

Now, $S = \{(x_1, y_1), \dots, (x_m, y_m)\}$, $0 \le x_1 < \dots < x_m < n$. Let w pass through all points of S, then $\mathbf{1}_{(W=w)}\tau(S) = \mathbf{1}_{(W=w)}\prod_{k=1}^m \big(w(x_k+1) - y_k\big)$ and $\mathbb{E}\left(\tau(S)\mathbf{1}_{(W=w)}\right) = 2^{-n}\prod_{k=1}^m \big(w(x_k+1) - y_k\big)$. Therefore $\hat\xi(S) = n^{-1/2}2^{-n}\sum_w w(n)\prod\big(w(x_k+1) - y_k\big)$, the sum being taken over w that satisfy $w(x_k) = y_k$ for all k. The set of such w is a product of $m+1$ sets, corresponding to the partition of $[0, n)$ into $[0, x_1)$, $[x_1, x_2)$, ..., $[x_{m-1}, x_m)$, $[x_m, n)$, and the sum over w decomposes into a product of $m+1$ sums. The first sum is equal to $\binom{x_1}{(x_1-y_1)/2} = 2^{x_1}p(x_1, y_1)$. The second sum, containing $w(x_1+1) - y_1$, is equal to $2^{x_2-(x_1+1)}p\big(x_2 - (x_1+1), y_2 - (y_1+1)\big) - 2^{x_2-(x_1+1)}p\big(x_2 - (x_1+1), y_2 - (y_1-1)\big)$. And so on. The last sum, containing $w(x_m+1) - y_m$, but also $w(n)$, is equal to $2^{n-(x_m+1)}(y_m+1) - 2^{n-(x_m+1)}(y_m-1) = 2^{n-x_m}$. So,

$$
\begin{aligned}
&\hat\xi(S) = \frac{1}{\sqrt{n}} \cdot 2^{-n} \cdot 2^{x_1}p(x_1, y_1) \times \\
&\left(\prod_{k=1}^{m-1} 2^{x_{k+1}-x_k-1}\big(p(x_{k+1} - x_k - 1, y_{k+1} - y_k - 1) - p(\dots, y_{k+1} - y_k + 1)\big)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times 2^{n-x_m} = \frac{1}{\sqrt{n}}p(x_1, y_1)q(S)\,.
\end{aligned}
$$

□

The following general fact holds for arbitrary I and ξ.

2.1 Lemma Let I be a finite set, $\big(\tau_i\big)_{i\in I}$ be independent random signs ($\mathbb{E}\tau_i = 0$), and $\xi = \sum_{S\subset I}\hat\xi(S)\tau(S)$, where $\tau(S) = \prod_{i\in S}\tau_i$. Then for any $E \subset I$ and $T \subset I \setminus E$,

$$
\sum_{S\subset E}\big|\hat\xi(S\cup T)\big|^2 = \mathbb{E}\,\big|\,\mathbb{E}\left(\tau(T)\xi \,\big|\, \tau|_E\right)\big|^2
$$

(the conditional expectation $\mathbb{E}\left(\dots \,\middle|\, \tau|_E \right)$ is taken w.r.t. all $\tau(i)$ for $i \in E$).

PROOF. We have

$$\xi = \sum_{S_1 \subset E} \sum_{S_2 \subset I \setminus E} \hat{\xi}(S_1 \cup S_2) \tau(S_1) \tau(S_2) \,,$$

therefore

$$\mathbb{E}\left(\tau(T)\xi \,\middle|\, \tau|_E \right) = \sum_{S_1 \subset E} \tau(S_1) \sum_{S_2 \subset I \setminus E} \hat{\xi}(S_1 \cup S_2) \, \mathbb{E}\left(\tau(T)\tau(S_2) \,\middle|\, \tau|_E \right) .$$

However, $\mathbb{E}\left(\tau(T)\tau(S_2) \,\middle|\, \tau|_E \right) = \mathbb{E}\left(\tau(T)\tau(S_2) \right) = 0$ unless $S_2 = T$. So, $\mathbb{E}\left(\tau(T)\xi \,\middle|\, \tau|_E \right) = \sum_{S_1 \subset E} \tau(S_1)\hat{\xi}(S_1 \cup T)$ and $\mathbb{E}\, |\dots|^2 = \sum_{S_1 \subset E} |\hat{\xi}(S_1 \cup T)|^2$. $\square$

We return to special I and ξ introduced in Sect. 1. Recall the projection R^{random} of the random set S^{random}. It is never empty. The following result describes the (uniform, in fact) distribution of the maximal element of R^{random}.

2.2 Lemma $\mathbb{P}\left(R^{\text{random}} \subset [0,k] \right) = \dfrac{k+1}{n}$ for $k = 0, \dots, n-1$.

PROOF. Lemma 2.1 for $E = \{(x,y) \in I : x \le k\}$ and $T = \emptyset$ gives $\mathbb{P}\left(R^{\text{random}} \subset [0,k] \right) = \mathbb{E} \left| \mathbb{E}\left(\xi \,\middle|\, \tau|_E \right) \right|^2$. However, $\mathbb{E}\left(W(n) \,\middle|\, \tau|_E \right) = W(k+1)$, thus, $\mathbb{P}\left(R^{\text{random}} \subset [0,k] \right) = \mathbb{E}\, |n^{-1/2} W(k+1)|^2 = \frac{k+1}{n}$. $\square$

Recall $q(S)$ defined in Prop. 1.1 for S of the form (1.2).

2.3 Lemma Let $k \in \{0, \dots, n-2\}$, $E = \{(x,y) \in I : x \le k\}$, and $S \subset I \setminus E$ be of the form (1.2), then

$$\mathbb{E}\left(\tau(S)\xi \,\middle|\, \tau|_E \right) = \frac{1}{\sqrt{n}} p\big(x_1 - (k+1), y_1 - W(k+1) \big) q(S)$$

(x_1, y_1 being defined by (1.2)).

PROOF. Similarly to the proof of Prop. 1.1, but now $w(0), \dots, w(k+1)$ are fixed, and the sum is taken over $w(k+2), \dots, w(n)$.

2.4 Lemma Let k, E, S be as in Lemma 2.3, then

$$\mathbb{P}\left(S^{\text{random}} \setminus E = S \right) = \frac{1}{n} \bigg(\sum_y p(k+1, y) p^2(x_1 - k - 1, y_1 - y) \bigg) q^2(S) \,.$$

PROOF. Lemma 2.1 gives

$$\mathbb{P}\left(S^{\mathrm{random}} \setminus E = S \right) = \mathbb{E} \left| \mathbb{E} \left(\tau(S)\xi \,\middle|\, \tau|_E \right) \right|^2 .$$

Lemma 2.3 gives

$$\mathbb{E} \left(\tau(S)\xi \,\middle|\, \tau|_E \right) = \frac{1}{\sqrt{n}} p(x_1 - (k+1), y_1 - W(k+1)) q(S) .$$

So, $\mathbb{P}\left(S^{\mathrm{random}} \setminus E = S \right) = \mathbb{E} |\dots|^2 = \frac{1}{n} q^2(S)\, \mathbb{E}\, p^2(x_1-(k+1), y_1-W(k+1))$. It remains to note that $\mathbb{E}\, p^2(x_1-(k+1), y_1-W(k+1)) = \sum_y p^2(x_1-k-1, y_1-y) p(k+1,y)$. $\square$

Given a set $S \subset I$ of the form (1.2), we introduce the set $S^{\updownarrow}$ of all vertical shifts of S. That is, for $S = \{(x_1,y_1),\dots,(x_m,y_m)\}$, $S^{\updownarrow}$ consists of the sets $\{(x_1,y_1+\delta),\dots,(x_m,y_m+\delta)\}$, where δ satisfies $y_1+\delta \in \{-x_1, -x_1+2, \dots, x_1-2, x_1\}$.

2.5 Lemma Let k, E, S be as in Lemma 2.3, then

$$\mathbb{P}\left(S^{\mathrm{random}} \setminus E \in S^{\updownarrow} \right) = \frac{1}{n} p\big(2(x_1-k-1), 0\big) q^2(S) .$$

PROOF. By Lemma 2.4, the probability is

$$\sum_{y_1} \frac{1}{n} \Bigg(\sum_y p(k+1,y) p^2(x_1-k-1, y_1-y) \Bigg) q^2(S) ,$$

since $q(\cdot)$ is shift-invariant. However,

$$\sum_{y,y_1} p(k+1,y) p^2(x_1-k-1, y_1-y) = \\ = \Bigg(\sum_y p(k+1,y) \Bigg) \cdot \Bigg(\sum_y p^2(x_1-k-1,y) \Bigg) = \\ = \sum_y p^2(x_1-k-1,y) = p\big(2(x_1-k-1),0\big) ;$$

the latter equality is a well-known property of binomial coefficients, see for instance [2, II.12.11]. $\square$

2.6 Lemma Let $0 \le k < x_1 < \dots < x_m < n$, then

$$\mathbb{P}\left(R^{\mathrm{random}} \cap [k, x_1) = \emptyset \,\middle|\, R^{\mathrm{random}} \cap [x_1, n) = \{x_1, \dots, x_m\} \right) = \\ = p(2(x_1-k), 0) .$$

Proof. The event $R^{\text{random}} \cap [x_1, n) = \{x_1, \dots, x_m\}$ is the union of disjoint events of the form $S^{\text{random}} \setminus E_1 \in S^{\updownarrow}$, where $E_1 = \{(x, y) \in I : x < x_1\}$ and $S \subset I \setminus E_1$, $S = \{(x_1, y_1), \dots, (x_m, y_m)\}$ for some $y_1, \dots, y_m$ such that S satisfies (1.2). Given such S, we use Lemma 2.5, its k being our $x_1 - 1$; we get

$$\mathbb{P}\left(S^{\text{random}} \setminus E_1 \in S^{\updownarrow} \right) = \frac{1}{n} q^2(S) ,$$

since $p(0, 0) = 1$.

We use Lemma 2.5 once again, its k being now equal to our $k - 1$; we get

$$\mathbb{P}\left(S^{\text{random}} \setminus E_2 \in S^{\updownarrow} \right) = \frac{1}{n} p\big(2(x_1 - k), 0\big) q^2(S) ,$$

where $E_2 = \{(x, y) \in I : x < k\}$.

Note that $S^{\text{random}} \setminus E_2 \in S^{\updownarrow}$ if and only if $S^{\text{random}} \setminus E_1 \in S^{\updownarrow}$ and $R^{\text{random}} \cap [k, x_1) = \emptyset$. Thus,

$$\mathbb{P}\left(R^{\text{random}} \cap [k, x_1) = \emptyset \,\middle|\, S^{\text{random}} \setminus E_1 \in S^{\updownarrow} \right) = \\ = \frac{\mathbb{P}\left(S^{\text{random}} \setminus E_2 \in S^{\updownarrow} \right)}{\mathbb{P}\left(S^{\text{random}} \setminus E_1 \in S^{\updownarrow} \right)} = p\big(2(x_1 - k), 0\big) .$$

The conditional probability does not depend on S. Summing over all S (one S within each equivalence class $S^{\updownarrow}$), we get

$$\mathbb{P}\left(R^{\text{random}} \cap [k, x_1) = \emptyset \,\middle|\, R^{\text{random}} \cap [x_1, n) = \{x_1, \dots, x_m\} \right) = \\ = p\big(2(x_1 - k), 0\big) .$$

$\square$

Proof of Prop. 1.3. We have

$$\mathbb{P}\left(R^{\text{random}} = \{x_1, \dots, x_m\} \right) = \\ = \mathbb{P}\left(R^{\text{random}} \cap [x_m, n) = \{x_m\} \right) \times \\ \times \mathbb{P}\left(R^{\text{random}} \cap [x_{m-1}, x_m) = \{x_{m-1}\} \,\middle|\, R^{\text{random}} \cap [x_m, n) = \{x_m\} \right) \cdot \ldots \times \\ \times \mathbb{P}\left(R^{\text{random}} \cap [x_1, x_2) = \{x_1\} \,\middle|\, R^{\text{random}} \cap [x_2, n) = \{x_2, \dots, x_m\} \right) \times \\ \times \mathbb{P}\left(R^{\text{random}} \cap [0, x_1) = \emptyset \,\middle|\, R^{\text{random}} \cap [x_1, n) = \{x_1, \dots, x_m\} \right) .$$

The first factor is

$$\mathbb{P}\left(R^{\text{random}} \cap [x_m, n) = \{x_m\} \right) = \\ = \mathbb{P}\left(R^{\text{random}} \subset [0, x_m] \right) - \mathbb{P}\left(R^{\text{random}} \subset [0, x_m) \right) = \\ = \frac{x_m + 1}{n} - \frac{x_m}{n} = \frac{1}{n}$$

by Lemma 2.2. The second factor is

$$\mathbb{P}\left(R^{\text{random}} \cap [x_{m-1}, x_m) = \{x_{m-1}\} \,\middle|\, R^{\text{random}} \cap [x_m, n) = \{x_m\} \right) =$$
$$\mathbb{P}\left(R^{\text{random}} \cap (x_{m-1}, x_m) = \emptyset \,\middle|\, \dots \right) - \mathbb{P}\left(R^{\text{random}} \cap [x_{m-1}, x_m) = \emptyset \,\middle|\, \dots \right)$$
$$= p\big(2(x_m - x_{m-1} - 1), 0\big) - p\big(2(x_m - x_{m-1}), 0\big)$$

by Lemma 2.6. And so on. The last factor is

$$\mathbb{P}\left(R^{\text{random}} \cap [0, x_1) = \emptyset \,\middle|\, R^{\text{random}} \cap [x_1, n) = \{x_1, \dots, x_m\} \right) = p(2x_1, 0)\,.$$

So,

$$\mathbb{P}\left(R^{\text{random}} = \{x_1, \dots, x_m\} \right) = \frac{1}{n} \times$$
$$\times \left(\prod_{k=1}^{m-1} \big(p(2(x_{k+1} - x_k - 1), 0) - p(2(x_{k+1} - x_k), 0) \big) \right) \times$$
$$\times\, p(2x_1, 0)\,.$$

$\square$

References

[1] Itai Benjamini, Gil Kalai, Oded Schramm, "Noise sensitivity of Boolean functions and applications to percolation", math.PR/9811157.

[2] W. Feller, "An introduction to probability theory and its application", third edition, Wiley, NY 1968.

School of Mathematics, Tel Aviv Univ., Tel Aviv 69978, Israel
tsirel@math.tau.ac.il
http://math.tau.ac.il/∼tsirel/